

# Quantum metrology: why entanglement?

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We show why and when entanglement is needed for quantum-enhanced precision measurements, and which type of entanglement is useful. We give a simple, intuitive construction that shows how entanglement transforms parallel estimation strategies into sequential ones of same precision. We employ this argument to generalize conventional quantum metrology, to identify a class of noise whose effects can be easily managed, and to treat the case of indistinguishable probes (such as interferometry with light).

To measure a parameter  $\varphi$  that identifies some physical transformation  $U_\varphi$ , we can sample the transformation with a probe system and then measure the change in the state of the probe. To reduce the statistical error, we repeat the sampling  $N$  times and average the results. Quantum metrology encompasses a large class of estimation procedures where quantum effects such as entanglement and squeezing are used to enhance the precision of such measurement over what would be possible with classical resources [1, 2]. Such techniques are by now well established [3–10] and have produced impressive results [11–22]. Yet, the reason *why* entanglement is necessary has not been settled. Here we present a simple, intuitive argument on why and when entanglement is necessary and, additionally, use it to derive some new results in quantum metrology. Our argument shows how entanglement can convert a parallel-sampling strategy (where  $N$  probes sample the system in parallel) to a sequential strategy, which is known to achieve high precision, and viceversa. While the equivalence of parallel and sequential strategy was already shown in Ref. [11] using the theory of tensor networks, our construction is much simpler, as it is based on a trivial algebraic identity: this simplicity is key to give an intuition on what quantum metrology means, to obtain some new results, and to reobtain more easily some known ones: e.g., we show how one can easily generalize the conventional quantum metrology framework; which kinds of errors are simple to manage; how the theory accommodates indistinguishable probes (e.g. bosons for light interferometry); which types of entanglement are useful; how non-asymptotic results arise in quantum metrology; etc.

In the framework of quantum metrology we are given a “black box” that performs a unitary operation  $U_\varphi = e^{i\varphi H}$ , where  $\varphi$  is the parameter to be estimated and  $H$  is its generator of translations. What is the best precision with which we can determine  $\varphi$ ? Call  $|0\rangle$  and  $|1\rangle$  the states that correspond to the minimum and maximum eigenvalues of  $H$ . The existence of such states is guaranteed only in finite-dimensional Hilbert spaces, and we will restrict to this case here (see Refs. [23, 24] for some infinite-dimensional results). If we sample the system sequentially  $N$  times with a single probe (Fig. 1a),

we can perform a measurement of  $\varphi$  as follows: (a) prepare the probe in the state  $|+\rangle$  or  $|-\rangle$ ,  $|\pm\rangle \equiv |0\rangle \pm |1\rangle$  (we will neglect state normalizations); (b) evolve it using a sequence of  $N$  unitaries as  $U_\varphi^N|\pm\rangle = |0\rangle \pm e^{iN\varphi}|1\rangle$ ; (c) project the final state onto the initial one: the probability that they coincide is  $p(\varphi) = \cos(N\varphi/2)$  (Ramsey interferometry corresponds to the case  $N = 1$  [8]). By repeating the above procedure  $\nu$  times, the experimental probability  $p(\varphi)$ , whence  $\varphi$ , are inferred with an error that scales as  $1/(N\sqrt{\nu})$  [5, 25] where the factor  $\sqrt{\nu}$  comes from the central limit theorem and the factor  $N$  comes from the fact that the sequential procedure is equivalent to measuring the parameter  $N\varphi$ : an error  $\Delta$  on  $N\varphi$  corresponds to an error  $\Delta/N$  on  $\varphi$ . In contrast, if we prepare the  $N\nu$  probes in the state  $|+\rangle$  or  $|-\rangle$ , use them to sample the system in parallel and average the results (Fig. 1b), from the central limit theorem it follows that we achieve an error that asymptotically scales as  $1/\sqrt{N\nu}$ —the “standard quantum limit” (SQL). Surprisingly, if the  $N$  probes are prepared in an entangled state (Fig. 1c), the quantum Cramer-Rao bound can be used to show that one can asymptotically achieve the same  $1/N$  limit [3–5] of the sequential strategy—the “Heisenberg bound”. A typical quantum metrology experiment: (a) prepare  $N$  probes in the entangled state  $|0\rangle^N + |1\rangle^N$ ; (b) evolve it by applying  $U_\varphi$  to each probe in parallel:  $U^{\otimes N}(|0\rangle^N + |1\rangle^N) = |0\rangle^N + e^{iN\varphi}|1\rangle^N$ ; (c) project on the initial state as above to estimate the parameter  $N\varphi$ : as before, the probability is  $p(\varphi) = \cos(N\varphi/2)$ , whence  $\varphi$  can be estimated with an error that scales as  $1/(N\sqrt{\nu})$  as in the sequential case above. While this is well known, the reason why entanglement helps was not explored in depth: typically one would say that this is the effect of the “global” nature of the entangled state that cannot be factored into separate contributions for each probe. Here we give a simple construction that shows how one can convert a parallel-entangled strategy into a sequential one and viceversa, proving that this is a reason why they achieve the same error scaling. We show that classical correlation among the probes is insufficient.

*The argument:*— We start detailing our construction for  $N = 2$  and then extend it to arbitrary  $N$  by induction. Given an operator  $C = \sum_{ij} C_{ij}|i\rangle\langle j|$  ( $\{|i\rangle\}$  a basis), we

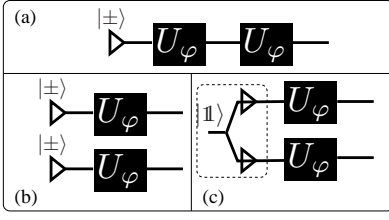


FIG. 1: Possible strategies for estimating the parameter  $\varphi$  that a unitary transformation  $U_\varphi$  (black boxes) encodes into a probe system (triangles). (a) Sequential strategies: a single probe samples the  $N$  boxes sequentially, effectively measuring  $N\varphi$  and achieving an error  $\propto N^{-1}$  on the estimation of  $\varphi$ . (b) Classical (unentangled) parallel strategies: we use  $N$  probes and average the measurement results: the central limit theorem tells us that the error on  $\varphi$  scales as  $N^{-1/2}$ . (c) Entangled strategy (quantum metrology): the  $N$  probes are prepared in a joint entangled state, but sample the black boxes separately: the error scales as  $N^{-1}$ .

can define a state  $|C\rangle \equiv \sum_{ij} C_{ij} |i\rangle |j\rangle$ . It can be used [26] for the following identity

$$A \otimes B |C\rangle = \sum_{ijkl} A_{ki} B_{lj} C_{ij} |k\rangle |l\rangle = |ACB^T\rangle, \quad (1)$$

where  $T$  indicates the transpose in the  $|i\rangle$  basis. Consider the quantum metrology protocol (Fig. 1c) for  $N = 2$ , and apply this identity twice:

$$(U_\varphi \otimes U_{\varphi'}) |\mathbb{1}\rangle = |U_\varphi U_{\varphi'}^T\rangle = (U_\varphi U_{\varphi'}^T \otimes \mathbb{1}) |\mathbb{1}\rangle, \quad (2)$$

where  $|\mathbb{1}\rangle = |00\rangle + |11\rangle$ , and we consider the general case where  $\varphi$  and  $\varphi'$  may differ. Choose  $|i\rangle$  as the eigenbasis of  $H$ , so  $U_\varphi = e^{i\varphi H}$  is diagonal, and  $U_\varphi^T = U_\varphi$ . Moreover, we can exploit the entanglement of  $|\mathbb{1}\rangle$  to measure the second system in the  $\{|+\rangle, |-\rangle\}$  basis (we consider only the subspace spanned by  $|0\rangle$  and  $|1\rangle$ ) and project the first system in the same state, since  $|00\rangle + |11\rangle = |++\rangle + |--\rangle$ . This means that after the measurement, the evolution in Eq. (2) can be written as  $U_\varphi U_{\varphi'} |\pm\rangle$ , where the initial state is  $|+\rangle$  or  $|-\rangle$  with probability  $1/2$ , depending on the measurement outcome. This argument shows how the parallel entangled strategy  $(U_\varphi \otimes U_{\varphi'}) |\mathbb{1}\rangle$  can be reduced to the sequential one  $U_\varphi U_{\varphi'} |\pm\rangle$  for  $N = 2$ , see Fig. 2.

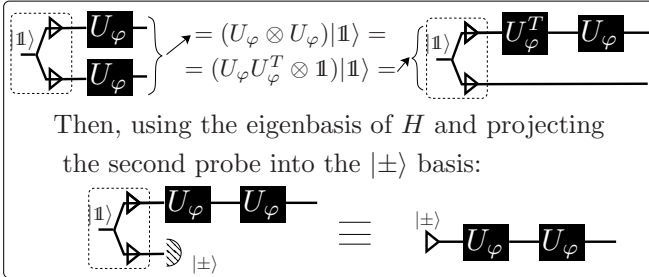


FIG. 2: Argument to transform a parallel entangled strategy into a sequential one for  $N = 2$ , see Eq. (2).

Why is entanglement necessary? We must require that perfect correlation is present in two complementary ba-

sis, the  $|0, 1\rangle$  basis (to ensure that  $U_\varphi$  is diagonal) and the  $|\pm\rangle$  basis (to ensure that the measurement on one probe projects the other in an appropriate initial state). An intuitive definition of entanglement is “correlation on a property that does not (cannot) yet exist”, namely, a perfect correlation on complementary properties such as the two bases above. In fact, suppose we request that there is correlation only on the  $|0, 1\rangle$  basis using the mixed state  $|00\rangle\langle 00| + |11\rangle\langle 11|$ , that can be written as an equally weighted mixture of  $|\mathbb{1}\rangle = |00\rangle + |11\rangle$  and  $|\sigma_z\rangle = |00\rangle - |11\rangle$ , each with probability  $1/2$ . We can use the construction of Eq. (2) obtaining a final state of the two probes given by  $(U_\varphi U_\varphi \otimes \mathbb{1}) |\mathbb{1}\rangle$  or  $(U_\varphi \sigma_z U_\varphi \otimes \mathbb{1}) |\mathbb{1}\rangle$  each with probability  $1/2$ . After the projection of the second probe into the  $|\pm\rangle$  basis, these become an equally weighted mixture of  $|0\rangle \pm e^{i2\varphi} |1\rangle$  and  $|0\rangle \mp e^{i2\varphi} |1\rangle$ , namely the state  $|0\rangle\langle 0| + |1\rangle\langle 1|$ , independent of  $\varphi$ . In contrast, suppose we request correlation only on the  $|\pm\rangle$  basis using the state  $|++\rangle\langle ++| + |--\rangle\langle --|$ , namely an equally weighted mixture of  $|\mathbb{1}\rangle = |++\rangle + |--\rangle$  and  $|\sigma_x\rangle = |++\rangle - |--\rangle$  (where  $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$  in the  $|0, 1\rangle$  basis). Again using Eq. (2), this is equivalent to having a final state of the two probes as  $(U_\varphi U_\varphi \otimes \mathbb{1}) |\mathbb{1}\rangle$  or  $(U_\varphi \sigma_x U_\varphi \otimes \mathbb{1}) |\mathbb{1}\rangle$  each with probability  $1/2$ , which, after the projection, become an equally weighted mixture of the four states  $|0\rangle \pm e^{i2\varphi} |1\rangle$  and  $|0\rangle \pm |1\rangle$ , again  $|0\rangle\langle 0| + |1\rangle\langle 1|$  independent of  $\varphi$ . [Here the SQL is recovered if one does not average over the projective-measurement result: that procedure is equivalent to the classical parallel strategy of Fig. 1b.]

We have shown that for  $N = 2$  we can easily convert a parallel entangled strategy into a sequential one, proving that the two have the same estimation precision. We now extend this to arbitrary  $N$  by induction: suppose (induction hypothesis) that the parallel entangled strategy  $\bigotimes_{j=1}^N U_{\varphi_j} (|0\rangle^N + |1\rangle^N)$  has the same estimation precision as the sequential strategy  $\prod_{j=1}^N U_{\varphi_j} |\pm\rangle$ , we need to prove that the same applies for  $N + 1$ . We can write

$$\begin{aligned} & \bigotimes_{j=1}^{N+1} U_{\varphi_j} (|0\rangle^{N+1} + |1\rangle^{N+1}) \\ &= \bigotimes_{j=1}^{N-1} U_{\varphi_j} \otimes \bar{U}_\theta (|0\rangle^{N-1} |\bar{0}\rangle + |1\rangle^{N-1} |\bar{1}\rangle), \end{aligned} \quad (3)$$

where  $|\bar{0}\rangle = |00\rangle$ ,  $|\bar{1}\rangle = |11\rangle$ , and

$$\begin{aligned} \bar{U}_\theta &\equiv U_{\varphi_N} \otimes U_{\varphi_{N+1}} = e^{i\lambda} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_N} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_{N+1}} \end{pmatrix} \\ &= e^{i\lambda} \begin{pmatrix} 1 & & \\ & e^{i\varphi_{N+1}} & \\ & & e^{i\varphi_N} \\ & & & e^{i(\varphi_N + \varphi_{N+1})} \end{pmatrix}, \end{aligned} \quad (4)$$

since  $U_{\varphi_j}$ 's are diagonal in the  $|0, 1\rangle$  basis. It follows that the subspace spanned by  $|\bar{0}\rangle = |00\rangle$  and  $|\bar{1}\rangle = |11\rangle$  is invariant for application of  $\bar{U}_\theta$ , which, when restricted

to such subspace, can be expressed as the  $2 \times 2$  matrix

$$U_\theta = e^{i\lambda} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\varphi_N + \varphi_{N+1})} \end{pmatrix} = U_{\varphi_N} U_{\varphi_{N+1}}. \quad (5)$$

Restricting ourselves to such invariant subspace, we can use the induction hypothesis to conclude that the last line of (3) will have the same estimation precision as  $\prod_{j=1}^{N-1} U_{\varphi_j} U_\theta |\pm\rangle = \prod_{j=1}^{N+1} U_{\varphi_j} |\pm\rangle$ , which proves the inductive step, see Fig. 3. [Note that in this proof it is not necessary to require that the matrix  $\bar{U}_\theta$  is diagonal, but only that it is invariant for the  $|0\rangle, |1\rangle$  subspace: an anti-diagonal tensor product would work.]

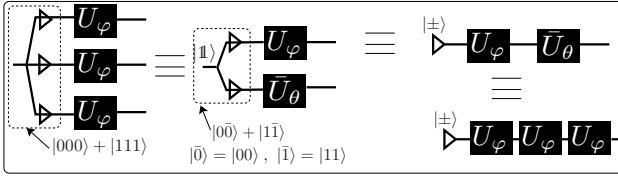


FIG. 3: Extension of the argument of Fig. 2 to arbitrary  $N$ .

*Consequences:*— We now explore some of the consequences that can be derived from our argument.

(a) Useful entanglement: Up to now we have used as entangled initial state the GHZ,  $(|0\rangle^N + |1\rangle^N)$ , as it optimizes the quantum Cramer-Rao bound. Our construction shows that entanglement of the type  $|\Psi_N\rangle = |0\rangle^N + e^{i\lambda}|1\rangle^N$  (with real  $\lambda$ ) is the only useful one (with the appropriate amendments when probes are indistinguishable, see below). In fact, suppose we use an arbitrary entangled state  $|E\rangle$  in Eq. (2), it becomes  $(e^{i\varphi H} \otimes e^{i\varphi H})|E\rangle = (e^{i\varphi H} E e^{i\varphi H} \otimes \mathbb{1})|\mathbb{1}\rangle$ , where we want the right hand term to correspond to a sequential strategy  $e^{2i\varphi H}(|0\rangle \pm e^{i\lambda}|1\rangle)$  (the sequential strategy works iff the initial state is of the form  $|0\rangle \pm e^{i\lambda}|1\rangle$  even though for simplicity we considered only the cases  $|\pm\rangle$  above). Namely, we require that  $e^{i\varphi H} E e^{i\varphi H} |\pm\rangle = e^{2i\varphi H}(|0\rangle \pm e^{i\lambda}|1\rangle)$ , which implies that  $E = |0\rangle\langle 0| + e^{i\lambda}|1\rangle\langle 1|$ , since  $H$  is diagonal in the  $|0, 1\rangle$  basis. Namely, we need a state  $|E\rangle = |\Psi_2\rangle$ . This argument, and the fact that achieving the quantum Cramer-Rao bound requires a superposition of global maximum and minimum eigenstates [5], shows that quantum metrology *requires* entanglement of the form  $|\Psi_N\rangle$ .

(b) Noise resilience: using our framework, we can detail a class of noise that is simple to handle (in general, the treatment of quantum metrology in the presence of noise is highly nontrivial [15–17, 27–29]). Consider the entangled parallel strategy for  $N = 2$  and suppose that the two probes are subject to arbitrary noise maps described by Kraus operators  $A_k$  and  $B_k$  respectively. We can still convert a parallel strategy into a sequential one

along the lines of Eq. (2) as

$$\begin{aligned} \sum_{jk} (A_k \otimes B_j) |\mathbb{1}\rangle\langle \mathbb{1}| (A_k^\dagger \otimes B_j^\dagger) &= \sum_{jk} |A_k B_j^T\rangle\langle A_k^* B_j^\dagger| \\ &= \sum_{jk} (A_k B_j^T \otimes \mathbb{1}) |\mathbb{1}\rangle\langle \mathbb{1}| (B_j^* A_k^\dagger \otimes \mathbb{1}), \end{aligned} \quad (6)$$

i.e. a noise map only on the first probe if  $\sum_j B_j B_j^\dagger = \mathbb{1}$  (a unital map, as when  $B_j \propto \sigma_\alpha$ , the Pauli matrices), since then  $\sum_{jk} B_j^* A_k^\dagger A_k B_j^T = \mathbb{1}$ , with  $B^* \equiv (B^T)^\dagger$ . So the first step of the induction can be applied also to unital maps in addition to the unitaries we discussed above. However, the extension to  $N > 2$  fails in general, as there is typically no invariant subspace to construct a reduced map as the one in (5): one must require that  $A_k \otimes B_j$  is a diagonal or an anti-diagonal matrix for all  $k, j$ . An example of the latter is the bit flip with phase noise where the Kraus operators are  $A_0 \propto \sigma_x$  and  $A_1 \propto \sigma_y$ . An example of the former (in addition to the noiseless case) is phase flip noise (dephasing) with Kraus operators  $A_0 \propto \mathbb{1}$  and  $A_1 \propto \sigma_z$ , which was already identified in [11] as a noise model whose parallel and sequential performance match. It was known for a long time [17] that an arbitrary small amount of dephasing would ruin any advantage for frequency measurement [15, 18–20] whereas the same identical noise does not affect phase estimation as much [2, 20]. Our construction explains this curious performance: since the entangled parallel strategy is (for this noise model) equivalent to the sequential one, it means that the entangled strategy’s performance is equivalent to one that takes  $N$  time as long: namely, its noise sensitivity is  $N$  times faster [17]. While this is not a problem for estimating a phase (which can be done in arbitrarily short time), it is a problem for estimating frequency which requires a measurement duration that voids any advantage from entanglement [2, 17, 20]. The calculation of the quantum Cramer-Rao bound for frequency-estimation on a single qubit subject to dephasing (the optimal sequential strategy) is given in Ref. [20], whence one can see it matches with the optimal entangled strategy (although that was not observed there). Our construction thus identifies some noise models whose effect is equivalent for the sequential and parallel strategies: the unital maps (in the  $|0\rangle, |1\rangle$  invariant subspace) where all Kraus operators are either diagonal or anti-diagonal, as the examples above. It is then simple using our construction to find the states that optimize the precision in the presence of these classes of noise.

(c) Indistinguishable probes: the above discussion focused on the case of distinguishable probes, where a tensor product structure (TPS) is clear: each probe is a system with its own Hilbert space. If the probes are indistinguishable particles (e.g. in optical interferometry, where typically one considers single-photon probes), such structure is absent, and one typically resorts to the Fock space that enforces the symmetry properties required (symme-

try for exchange of Bosons and anti-symmetry for exchange of Fermions). In the absence of a TPS, the very concept of entanglement is ill defined [30, 31], and it is known that entanglement is not necessary to beat the SQL in interferometry. For example, the unentangled “N0” state  $|N\rangle + |\emptyset\rangle$ , where  $|N\rangle$  is a Fock state of  $N$  photons and  $|\emptyset\rangle$  is the vacuum state (we restrict ourselves to a finite-dimensional subspace of the radiation Hilbert space) can achieve a phase sensitivity beyond the SQL: its evolution through  $e^{i\varphi a^\dagger a}$  ( $a$  being the annihilation operator of the mode) yields  $|\emptyset\rangle + e^{iN\varphi}|N\rangle$ , as in the sequential or entangled strategies of Figs. 1a,c. Because such a state cannot be attained in the presence of super-selection rules [32] and it only samples an absolute phase, in quantum-enhanced interferometry one typically considers the two mode “N00N” state [8]  $|N, \emptyset\rangle + |\emptyset, N\rangle$ , which samples a relative phase between the two modes  $a, b$  of the interferometer through the evolution  $e^{i\varphi(a^\dagger a - b^\dagger b)}$ . While clearly  $|N, \emptyset\rangle + |\emptyset, N\rangle$  has mode entanglement, the state  $|N\rangle + |\emptyset\rangle$  has no entanglement. Yet, they can both be easily cast in the framework we have used up to now, by recalling that for Boson probes, one needs to consider only the subspace of the global Hilbert space that is symmetric for exchange of the probes, which can be done by moving to the Fock space [33]: in each mode we have that the tensor product of minimum and maximum eigenstates  $|0\rangle^N$  and  $|1\rangle^N$  is symmetrized as  $|\emptyset\rangle$  and  $|N\rangle$ , respectively. Analogously, in the two mode case, the symmetrized minimum and maximum eigenstates for the two-mode Hamiltonian  $a^\dagger a - b^\dagger b$  restricted to the  $N$ -photon subspace is  $|\emptyset, N\rangle$  and  $|N, \emptyset\rangle$ , respectively (the symmetrization must be performed in each mode separately, since photons in different modes are distinguishable). It is hence clear that N00N state interferometry falls within the framework described here. In particular, our construction shows that N00N state interferometry is indeed equivalent to multipass interferometry, as was pointed out previously [14, 25]. The same trick of considering the (anti-symmetric) Fock space can also be applied to the Fermionic case, although the Pauli exclusion principle makes Fermionic probes unattractive. [One may wonder whether the conventional theory of distinguishable probes is ever applicable, since clearly all probes will be ultimately be composed of indistinguishable particles. We remind that indistinguishable particles are subject to specific symmetries only regarding their global state. Typically one uses only certain degrees of freedom of the probe system for measurement purposes and the remaining degrees of freedom (spin, position, etc.) can be used to enforce the required symmetry.]

(d) Extension of quantum metrology: Conventionally, in quantum metrology the transformation that acts on the probes takes the form  $U_\varphi = e^{i\varphi H}$ . Our construction makes it easy to generalize this to a more general  $U'_\varphi = W e^{i\varphi H} V$ , where  $W$  and  $V$  are unitaries in-

dependent of  $\varphi$ . In this case the sequential strategy cannot be a simple iteration of  $U'_\varphi$ , as that will not typically increase the parameter  $\varphi$  by  $N$  times (e.g., when  $W = \mathbb{1}$ ,  $V = \sigma_x$  we find that  $U'^2_\varphi \propto \mathbb{1}$ ), but  $e^{iN\varphi H} = (W^\dagger U'_\varphi V^\dagger)^N$ . This suggests that a generalized entangled parallel strategy consists in the evolution  $(W^\dagger U'_\varphi V^\dagger)^{\otimes N}(|0\rangle^N + |1\rangle^N)$ , which corresponds to the sequential strategy  $(W^\dagger U'_\varphi V^\dagger)^N|\pm\rangle$ .

Is our framework general? The only hypothesis made so far is that there exists eigenvectors  $|0\rangle$  and  $|1\rangle$  for  $H$  connected to the maximum and minimum eigenvalues. This requirement, certainly true in finite-dimensional Hilbert spaces, implies that the states  $|+\rangle$  and  $|-\rangle$  (for the sequential strategy) and the GHZ state  $|0\rangle^N + |1\rangle^N$  (for the parallel strategy) are ones with maximum spread  $\Delta H$ , that saturate the quantum Cramer-Rao bound [3–5]. Our framework is thus general for finite-dimensional systems, and allows one to construct new quantum metrology protocols [5]: all that is needed is a GHZ-type state with a superposition of maximum and minimum eigenstates of  $H$  to achieve a quantum metrology protocol to estimate the parameter  $\varphi$  of which  $H$  is the generator of translations.

(e) The non-asymptotic case: all results given up to now (and most of the literature) refer to the asymptotic case where the number of repetitions  $\nu \rightarrow \infty$ . However, thanks to our construction, it is possible to adapt the non-asymptotic results [6, 14, 21, 34] obtained for the sequential estimation (by adapting the Kitaev phase-estimation algorithms) also to parallel entangled strategies. This equivalence was already pointed out in Ref. [6]: all these strategies employ multiple rounds of sequential iterations or, equivalently, multiple rounds of N00N states with different  $N$ . A completely different strategy is given in [22].

(f) Bandwidth-noise tradeoff: our construction clarifies that the true power of entanglement in quantum metrology is in allowing one to achieve the same  $\sqrt{N}$  precision enhancement of the sequential strategy, while keeping the same  $N$ -fold sampling-time gain of the parallel strategy over the sequential one [35]: high precision in short time.

*Conclusions:*— We presented a simple construction that shows why entanglement is necessary in quantum metrology: to transform a parallel strategy into a sequential one, we need perfect correlation in complementary bases: both in the eigenbasis  $|0, 1\rangle$  of the generator  $H$  and in its complementary  $|\pm\rangle$  basis. Such correlation requires entanglement. We also presented some applications: our construction easily derives some new and some known results in quantum metrology.

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